[Kavitha, 6(7): July 2019]
DOI- 10.5281/zenodo.3351440

ISSN 2348 – 8034
Impact Factor- 5.070

GLOBAL JOURNAL OF ENGINEERING SCIENCE AND RESEARCHES

ON THE POSITIVE PELL EQUATION

$$x^2 = 6y^2 + 3^t$$

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ABSTRACT

The binary quadratic Diophantine equation represented by the positive pellian $x^2 = 6y^2 + 3^t$ is analyzed for its non-zero distinct solutions. A few interesting relations among the solutions are given. Further, employing the solutions of the above hyperbola, we have obtained solutions of other choices of hyperbolas, parabolas and Pythagorean triangle.

***Keywords**: Binary quadratic, Hyperbola, Parabola, Integral solutions, Pell equation.*

I. INTRODUCTION

The binary quadratic equations of the form $y^2 = Dx^2 + 1$ where D is non-square positive integer has been selected by various mathematicians for its non-trivial integer solutions when D takes different integral values [1-4]. For an extensive review of various problems, one may refer [5-10]. In this communication, yet another interesting equation given by $x^2 = 6y^2 + 3^t$ is considered and infinitely many integer solutions are obtained. A few interesting properties among the solutions are presented.

II. METHOD OF ANALYSIS

The positive Pell equation representing hyperbola under consideration is,

$$x^2 = 6y^2 + 3^t \tag{1}$$

The smallest positive integer solutions of (1) are,

$x_0 = 3.3^{t-1}$, $y_0 = 1.3^{t-1}$

The pellian equation is

$$x^2 = 6y^2 + 1 \tag{2}$$

The initial solution of pellian equation is

$\tilde{x}_0 = 5$, $\tilde{y}_0 = 2$

The general solution (x_n, y_n) of (2) is given by,

$$\tilde{x}_n = \frac{1}{2} f_n, \quad \tilde{y}_n = \frac{1}{2\sqrt{6}} g_n$$

Where,

$$f_n = \left(5 + 2\sqrt{6}\right)^{n+1} + \left(5 - 2\sqrt{6}\right)^{n+1}$$

$$g_n = \left(5+2\sqrt{6}\right)^{n+1} - \left(5-2\sqrt{6}\right)^{n+1}$$

Applying Brahmagupta lemma between (x_0, y_0) and $(\tilde{x}_n, \tilde{y}_n)$ the other integer solution of (1) are given by,

$$x_{n+1} = \frac{3.3^{t-1}}{2} f_n + \frac{6.3^{t-1}}{2\sqrt{6}} g_n$$

$$y_{n+1} = \frac{3^{t-1}}{2} f_n + \frac{3.3^{t-1}}{2\sqrt{6}} g_n$$

The recurrence relation satisfied by the solution x and y are given by,

$$x_{n+3} - 10x_{n+2} + x_{n+1} = 0$$

$$y_{n+3} - 10y_{n+2} + y_{n+1} = 0 \quad n=0,1,2,3\ldots..$$

Some numerical examples of x and y satisfying (1) are given in the Table 1 below,

Table 1: Examples

n	x_n	y_n
0	3.3^{t-1}	1.3^{t-1}
1	27.3^{t-1}	11.3^{t-1}
2	267.3^{t-1}	109.3^{t-1}
3	2643.3^{t-1}	1079.3^{t-1}

From the above table, we observe some interesting relations among the solutions which are presented below.

Both x_n and y_n values are odd.

Each of the following expression is a nasty number

$$\frac{6}{3^t}\left[2.3^t + 11x_{2n+2} - x_{2n+3}\right]$$

$$\frac{3}{10.3^t}\left[40.3^t + 218x_{2n+2} - 2x_{2n+4}\right]$$

$$\frac{6}{5.3^t}\left[10.3^t + 54x_{2n+2} - 12y_{2n+3}\right]$$

$$\frac{6}{49.3^t}\left[98.3^t + 534x_{2n+2} - 12y_{2n+4}\right]$$

$$\frac{6}{5.3^t}\left[10.3^t + 6x_{2n+3} - 132y_{2n+2}\right]$$

$$\frac{6}{49.3^t}\left[98.3^t + 6x_{2n+4} - 1308y_{2n+2}\right]$$

$$\frac{6}{3^t}\left[2.3^t + 3y_{2n+3} - 27y_{2n+2}\right]$$

$$\frac{3}{10.3^t}\left[40.3^t + 6y_{2n+4} - 534y_{2n+4}\right]$$

$$\frac{1}{3^t}\left[12.3^t + 654x_{2n+3} - 66x_{2n+4}\right]$$

$$\frac{6}{3^t}\left[2.3^t + 54x_{2n+3} - 132y_{2n+3}\right]$$

$$\frac{6}{5.3^t}\left[10.3^t + 534x_{2n+3} - 132y_{2n+4}\right]$$

$$\frac{6}{5.3^t}\left[10.3^t + 54x_{2n+4} - 1308y_{2n+3}\right]$$

$$\frac{6}{3^t}\left[2.3^t + 534x_{2n+4} - 1308y_{2n+4}\right]$$

$$\frac{6}{3^t}\left[2.3^t + 27y_{2n+4} - 267y_{2n+3}\right]$$

Each of the following expressions is a cubical integer.

$$\frac{1}{3^t}\left[11x_{3n+3} - x_{3n+4} + 33x_{n+1} - 3x_{n+2}\right]$$

$$\frac{1}{10.3^t}\left[109x_{3n+3} - x_{3n+5} + 327x_{n+1} - 3x_{n+2}\right]$$

$$\frac{1}{5.3^t}\left[54x_{3n+3} - 12y_{3n+4} + 162x_{n+1} - 36y_{n+2}\right]$$

$$\frac{1}{49.3^t}\left[534x_{3n+3} - 12y_{3n+5} + 1602x_{n+1} - 36y_{n+3}\right]$$

$$\frac{1}{5.3^t}\left[6x_{3n+4} - 132y_{3n+3} + 18x_{n+2} - 396y_{n+1}\right]$$

$$\frac{1}{49.3^t}\left[6x_{3n+5} - 1308y_{3n+3} + 18x_{n+2} - 3924y_{n+1}\right]$$

$$\frac{1}{3^t}\left[3y_{3n+4} - 27y_{3n+3} + 9y_{n+2} - 81y_{n+1}\right]$$

$$\frac{1}{10.3^t}\left[3y_{3n+5} - 267y_{3n+3} + 9y_{n+2} - 801y_{n+1}\right]$$

$$\frac{1}{3^t}\left[109x_{3n+4} - 11x_{3n+5} + 327x_{n+2} - 33x_{n+2}\right]$$

$$\frac{1}{3^t}\left[54x_{3n+4} - 132y_{3n+4} + 162x_{n+2} - 396y_{n+2}\right]$$

$$\frac{1}{5.3^t}\left[534x_{3n+4} - 132y_{3n+5} + 1602x_{n+2} - 396y_{n+3}\right]$$

$$\frac{1}{5.3^t}\left[54x_{3n+5}-1308y_{3n+4}+162x_{n+3}-3924y_{n+2}\right]$$

$$\frac{1}{3^t}\left[534x_{3n+5}-1308y_{3n+5}+1602x_{n+3}-3924y_{n+3}\right]$$

$$\frac{1}{3^t}\left[27y_{3n+5}-267y_{3n+4}+81y_{n+3}-801y_{n+2}\right]$$

Each of the following expressions is a biquadratic integer.

$$\frac{1}{3^t}\left[11x_{4n+4}-x_{4n+5}+44x_{2n+2}-4x_{2n+3}+6.3^t\right]$$

$$\frac{1}{10.3^t}\left[109x_{4n+4}-x_{4n+4}+436x_{2n+2}-4x_{2n+4}+60.3^t\right]$$

$$\frac{1}{5.3^t}\left[54x_{4n+4}-12y_{4n+5}+270x_{2n+2}-60y_{2n+3}+30.3^t\right]$$

$$\frac{1}{49.3^t}\left[534x_{4n+4}-12y_{4n+6}+2136x_{2n+2}-48y_{2n+4}+294.3^t\right]$$

$$\frac{1}{5.3^t}\left[6x_{4n+5}-132y_{4n+4}+24x_{2n+3}-528y_{2n+2}+30.3^t\right]$$

$$\frac{1}{49.3^t}\left[6x_{4n+6}-1308y_{4n+4}+24x_{2n+4}-5232y_{2n+2}+294.3^t\right]$$

$$\frac{1}{3^t}\left[3y_{4n+5}-27y_{4n+4}+12y_{2n+3}-108y_{2n+2}+6.3^t\right]$$

$$\frac{1}{10.3^t}\left[3y_{4n+6}-267y_{4n+4}+12y_{2n+4}-1068y_{2n+1}+60.3^t\right]$$

$$\frac{1}{3^t}\left[109x_{4n+5}-11x_{4n+6}+436x_{2n+3}-44x_{2n+4}+6.3^t\right]$$

$$\frac{1}{3^t}\left[54x_{4n+5}-132y_{4n+5}+216x_{2n+3}-528y_{2n+3}+6.3^t\right]$$

$$\frac{1}{5.3^t}\left[534x_{4n+5}-132y_{4n+6}+2136x_{2n+3}-528y_{2n+4}+30.3^t\right]$$

$$\frac{1}{5.3^t}\left[54x_{4n+6}-1308y_{4n+5}+216x_{2n+4}-5232y_{2n+3}+30.3^t\right]$$

$$\frac{1}{3^t}\left[534x_{4n+6}-1308y_{4n+6}+2136x_{2n+4}-5232y_{2n+4}+6.3^t\right]$$

$$\frac{1}{3^t}\left[27y_{4n+6}-267y_{4n+5}+108y_{2n+4}+106y_{2n+3}+6.3^t\right]$$

ISSN 2348 – 8034
Impact Factor- 5.070

Each of the following expression is a quintic integer:

$$\frac{1}{3^t}\left[6x_{5n+5}-12y_{5n+5}+30x_{3n+3}-60y_{3n+3}+60x_{n+1}-120y_{n+1}\right]$$

$$\frac{1}{3^t}\left[11x_{5n+5}-x_{5n+6}+55x_{3n+3}-5x_{3n+4}+110x_{n+1}-10x_{n+2}\right]$$

$$\frac{1}{5.3^t}\left[109x_{5n+5}-x_{5n+7}+545x_{3n+3}-5x_{3n+5}+1090x_{n+1}-10x_{n+2}\right]$$

$$\frac{1}{5.3^t}\left[54x_{5n+5}-12y_{5n+6}+270x_{3n+3}-60x_{3n+4}-540x_{n+1}-120y_{n+2}\right]$$

$$\frac{1}{49.3^t}\left[534x_{5n+5}-12y_{5n+7}+2670x_{3n+3}-60y_{3n+5}+5340x_{n+1}-120y_{n+3}\right]$$

$$\frac{1}{5.3^t}\left[6x_{5n+6}-132y_{5n+5}+30x_{3n+4}-660y_{3n+3}+60x_{n+2}-1320y_{n+1}\right]$$

$$\frac{1}{49.3^t}\left[6x_{5n+7}-1308y_{5n+5}+30x_{3n+5}-6540y_{3n+3}+60x_{n+2}-13080y_{n+1}\right]$$

$$\frac{1}{3^t}\left[3y_{5n+6}-27y_{5n+5}+15y_{3n+4}-135y_{3n+3}+30y_{n+2}-270y_{n+1}\right]$$

$$\frac{1}{10.3^t}\left[3y_{5n+7}-267y_{5n+5}+15y_{3n+5}-1335y_{3n+3}+30y_{n+2}-2670y_{n+1}\right]$$

$$\frac{1}{3^t}\left[109x_{5n+6}-11x_{5n+7}+545x_{3n+4}-55x_{3n+5}+1090x_{n+2}-110x_{n+3}\right]$$

$$\frac{1}{3^t}\left[54x_{5n+6}-132y_{5n+6}+270x_{3n+4}-660y_{3n+4}+540y_{n+2}-1320x_{n+2}\right]$$

$$\frac{1}{5.3^t}\left[534x_{5n+6}-132y_{5n+7}+2670x_{3n+4}-660y_{3n+5}+5340x_{n+2}-1320y_{n+3}\right]$$

$$\frac{1}{5.3^t}\left[54x_{5n+7}-1308y_{5n+6}+270x_{3n+5}-6540y_{3n+4}+540x_{n+3}-13080y_{n+2}\right]$$

$$\frac{1}{3^t}\left[27y_{5n+7}-267y_{5n+6}+135y_{3n+5}-1335y_{3n+4}+270y_{n+3}-2670y_{n+2}\right]$$

$$\frac{1}{3^t}\left[534x_{5n+7}-1308y_{5n+7}+2670x_{3n+5}-6540y_{3n+5}+5340x_{n+3}-13080y_{n+3}\right]$$

Relations among the solutions are given below:

$$x_{n+3}=49x_{n+1}+120y_{n+1}$$
$$y_{n+2}=2x_{n+1}+5y_{n+1}$$
$$y_{n+3}=20x_{n+1}+49y_{n+1}$$
$$x_{n+3}=10x_{n+2}-x_{n+1}$$

$12y_{n+2} = 5x_{n+2} - x_{n+1}$

$12y_{n+3} = 49x_{n+2} - 5x_{n+1}$

$24y_{n+2} = x_{n+3} - x_{n+1}$

$120y_{n+3} = 49x_{n+3} - x_{n+1}$

$5y_{n+3} = 49y_{n+2} + 2x_{n+1}$

$5x_{n+3} = 12y_{n+1} + 49x_{n+2}$

$5y_{n+2} = y_{n+1} + 2x_{n+2}$

$y_{n+3} = y_{n+1} + 4x_{n+2}$

$49y_{n+2} = 5y_{n+1} + 2x_{n+3}$

$49y_{n+3} = y_{n+1} + 20x_{n+3}$

$y_{n+3} = 10y_{n+2} - y_{n+1}$

$12y_{n+2} = x_{n+3} - 5x_{n+2}$

$12y_{n+3} = 5x_{n+3} - x_{n+2}$

$y_{n+3} = 5y_{n+2} + 2x_{n+2}$

$5y_{n+3} = y_{n+2} + 2x_{n+3}$

Remarkable Observation

Employing linear combinations among the solutions of (1), one may generate integer solutions for other choices of hyperbolas which are presented in table 2 below

Table 2: Hyperbola

S.NO	Hyperbola	(X,Y)
1	$X^2 - 6Y^2 = 4.9^t$	$(6x_{n+1} - 12y_{n+1}, 6y_{n+1} - 2x_{n+1})$
2	$X^2 - 6Y^2 = 4.9^t$	$(11x_{n+1} - x_{n+2}, 2x_{n+2} - 18x_{n+1})$
3	$X^2 - 6Y^2 = 1600.9^t$	$(218x_{n+1} - 2x_{n+3}, x_{n+3} - 89x_{n+1})$
4	$X^2 - 6Y^2 = 100.9^t$	$(54x_{n+1} - 12y_{n+2}, 6y_{n+2} - 22x_{n+1})$
5	$X^2 - 6Y^2 = 9604.9^t$	$(534x_{n+1} - 12y_{n+3}, 6y_{n+3} - 218x_{n+1})$
6	$X^2 - 6Y^2 = 100.9^t$	$(6x_{n+2} - 132y_{n+1}, 54y_{n+1} - 2x_{n+2})$
7	$X^2 - 6Y^2 = 9604.9^t$	$(6x_{n+3} - 1308y_{n+1}, 534y_{n+1} - 2x_{n+3})$
8	$X^2 - 6Y^2 = 4.9^t$	$(3y_{n+2} - 27y_{n+1}, 11y_{n+1} - y_{n+2})$
9	$X^2 - 6Y^2 = 1600.9^t$	$(6y_{n+3} - 534y_{n+1}, 218y_{n+1} - 2y_{n+3})$
10	$X^2 - 6Y^2 = 144.9^t$	$(654x_{n+2} - 66x_{n+3}, 27x_{n+3} - 267x_{n+2})$
11	$X^2 - 6Y^2 = 4.9^t$	$(54x_{n+2} - 132y_{n+2}, 54y_{n+2} - 22x_{n+2})$

12	$X^2-6Y^2=100.9^t$	$(534x_{n+2}-132y_{n+3}, 54y_{n+3}-218x_{n+2})$
13	$X^2-6Y^2=100.9^t$	$(54x_{n+3}-1308y_{n+2}, 534y_{n+2}-22x_{n+3})$
14	$X^2-6Y^2=4.9^t$	$(534x_{n+3}-1308y_{n+3}, 534y_{n+3}-218x_{n+3})$
15	$X^2-6Y^2=4.9^t$	$(27y_{n+3}-267y_{n+2}, 109y_{n+2}-11y_{n+3})$

Employing linear combination among the solutions of (1), one may generate integer solutions for other choices of parabolas which are presented in table 3 below

Table 3: Parabola

S.NO	Parabola	(X,Y)
1	$3^t X^2-6Y^2=4.9^t$	$(6x_{2n+2}-12y_{2n+2}+2.3^t, 6y_{n+1}-2x_{n+1})$
2	$3^t X^2-6Y^2=4.9^t$	$(11x_{2n+2}-x_{2n+3}+2.3^t, 2x_{n+2}-18x_{n+1})$
3	$20.3^t X^2-6Y^2=1600.9^t$	$(218x_{2n+2}-2x_{2n+4}, x_{n+3}-89x_{n+1})$
4	$5.3^t X^2-6Y^2=100.9^t$	$(54x_{2n+2}-12y_{2n+3}+10.3^t, 6y_{n+2}-22x_{n+1})$
5	$49.3^t X^2-6Y^2=9604.9^t$	$(534x_{2n+2}-12y_{2n+4}+98.3^t, 6y_{n+3}-218x_{n+1})$
6	$5.3^t X^2-6Y^2=100.9^t$	$(6x_{2n+3}-132y_{2n+2}+10.3^t, 54y_{n+1}-2x_{n+2})$
7	$49.3^t X^2-6Y^2=9604.9^t$	$(6x_{2n+4}-1308y_{2n+2}+98.3^t, 534y_{n+1}-2x_{n+3})$
8	$4.9^t X^2-6Y^2=4.9^t$	$(3y_{2n+3}-27y_{2n+2}+2.3^t, 11y_{n+1}-y_{n+2})$
9	$20.3^t X^2-6Y^2=1600.9^t$	$(6y_{2n+4}-534y_{2n+3}+40.3^t, 218y_{n+1}-2y_{n+3})$
10	$6.3^t X^2-6Y^2=144.9^t$	$(654x_{2n+3}-66x_{2n+4}+12.3^t, 27x_{n+3}-267x_{n+2})$

11	$3^t X^2 - 6Y^2 = 4.9^t$	$\begin{pmatrix} 54x_{2n+3} - 132y_{2n+3} + 2.3^t, \\ 54y_{n+2} - 22x_{n+2} \end{pmatrix}$
12	$5.3^t X^2 - 6Y^2 = 100.9^t$	$\begin{pmatrix} 534x_{2n+3} - 132y_{2n+4} + 10.3^t, \\ 54y_{n+3} - 218x_{n+2} \end{pmatrix}$
13	$5.3^t X^2 - 6Y^2 = 100.9^t$	$\begin{pmatrix} 54x_{2n+4} - 1308y_{2n+3} + 10.3^t, \\ 534y_{n+2} - 22x_{n+3} \end{pmatrix}$
14	$3^t X^2 - 6Y^2 = 4.9^t$	$\begin{pmatrix} 534x_{2n+4} - 1308y_{2n+4} + 2.3^t, \\ 534y_{n+3} - 218x_{n+3} \end{pmatrix}$
15	$3^t X^2 - 6Y^2 = 4.9^t$	$\begin{pmatrix} 27y_{2n+4} - 267y_{2n+3} + 2.3^t, \\ 109y_{n+2} - 11y_{n+3} \end{pmatrix}$

III. CONCLUSION

In this paper, we have presented infinitely many integer solutions for the positive Pell Equations $x^2 = 6y^2 + 3^t$ As the binary quadratic Diophantine equations are rich in variety, one may search for the other choices of Pell Equations and determine their integer solutions along with suitable properties.

REFERENCES

1. *L.E.Dickson, History of theory of number, Chelsa publishing company, vol-2, (1952) New York.*
2. *L.J.Mordel, Diophantine equations, Academic Press, (1969) New York.*
3. *S.J.Telang, Number Theory, Tata McGraw Hill Publishing company Limited,(2000) New Delhi.*
4. *D.M.Burton, Elementary Number Theory, Tata McGraw Hill Publishing company Limited,(2002) New Delhi.*
5. *M.A.Gopolan, s.vidhyalakshmi and A.Kavitha, on the integral solutions of the Binary quadratic Equation* $X^2 = 4(K^2 + 1)Y^2 + 4^t$ *, Bulletin of Mathematics and Statistics Research, 2(1)(2014)42-46.*
6. *S.Vidhyalakshmi, A.Kavitha and M.A.Gopalan, On the binary quadratic Diophantine equation* $x^2 - 3xy + y^2 + 33x = 0$*, International Journal of Scientific Engineering and Applied Science, 1(4) (2015) 222-225.*
7. *M.A.Gopalan, S.Vidhyalakshmi and A.Kavitha, observations on the Hyperbola* $10y^2 - 3x^2 = 13$*, Archimedes J.Math.,3(1) (2013) 31-34.*
8. *S.Vidhyalakshmi, A.Kavitha and M.A.Gopalan, Observations on the Hyperbola* $ax^2 - (a+1)y^2 = 3a - 1$*, Discovery, 4(10) (2013) 22-24.*
9. *S.Vidhyalakshmi, A.Kavitha and M.A.Gopalan, Integral points on the Hyperbola* $x^2 - 4xy + y^2 + 15x = 0$*, Diophantus J.Maths,1(7) (2014) 338-340.*
10. *M.A.Gopalan, S.Vidhyalakshmi and A.Kavitha, on the integral solutions Of the binary quadratic equation* $x^2 = 15y^2 - 11^t$ *, scholars journal of Engineering and technology, 2(2A) (2014) 156-158.*